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Numerical approximation of generalized Newtonian fluids using Powell–Sabin–Heindl elements: I. theoretical estimates

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SUMMARY

In this paper we consider the numerical approximation of steady and unsteady generalized Newtonian fluid flows using divergence free finite elements generated by the Powell–Sabin–Heindl elements. We derive *a priori* and *a posteriori* finite element error estimates and prove convergence of the method of successive approximations for the steady flow case. *A priori* error estimates of unsteady flows are also considered. These results provide a theoretical foundation and supporting numerical studies are to be provided in Part II. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: non-Newtonian fluids; divergence free elements; Powell–Sabin–Heindl bases; error estimates

1. INTRODUCTION

Shear thinning fluids are found in many important engineering and biological processes. Although the power law model is widely used in practice, in many instances this empirical law often does not fit the data well. Several other constitutive relations have been proposed in an attempt to better describe these fluids. Examples of these models include the power law, Carreau, Eyring, Oldroyd and Williamson models. For example, the extended Williamson fluid model has been shown to be capable of fitting experimental data well over a large range of shear and strain rates [1]. An Oldroyd-B model has been used in the study of blood flow [2]. This is our motivation for studying the class of generalized Newtonian fluid models in this paper.

In the numerical study of incompressible viscous flow, two main sources of difficulty are the need to preserve mass conservation and the demand to satisfy the inf–sup condition in order to achieve stability. These topics have been extensively studied for mixed finite element methods applied to Navier–Stokes problems. The mixed finite element method has also previously been employed in the study of the power law model and the Carreau model ([3–5].) Techniques

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developed in these studies allow the proper handling of non-linearity and as a consequence optimal energy error estimates for some practical elements are derived.

With the use of divergence free elements, mass conservation is built into the numerical model and the inf–sup condition is automatically satisfied. Such elements have been devised in the past, but in practice analysts have largely resisted using such elements mainly because of the inherent complexity in their construction and implementation. However, we will show in this paper that divergence free finite elements generated from Powell–Sabin–Heindl (PSH) elements, which we shall call curl(PSH) finite elements, are a viable family with good approximation properties for 2D generalized Newtonian flow models.

The choice of the curl(PSH) triangle element is motivated not only by the divergence free nature of the element, but also by our desire to keep the degree of the element as low as possible in view of the non-linearity in the partial differential equation that we need to solve. With approximation based on curl(PSH) elements, the non-linear coefficient becomes piecewise constant over each constituent subtriangle and this, in turn, leads to much easier construction of the corresponding non-linear algebraic system. Furthermore this piecewise constant property may be exploited in deriving *a posteriori* error estimates, as we show in Section 5.

The PSH element [6] is often referred to in the approximation literature as the Powell–Sabin 12-split element. There is also a 6-split version of the Powell–Sabin element [7]. With proper choice of the interior point, the result presented in this paper will apply to the 6-split element. Higher order elements with Powell–Sabin splits are also available, but even though the *a priori* estimates described in this paper would still be applicable, the use of these higher order elements is less appealing for the reasons noted above.

The structure of the treatment is as follows: first, in Section 2, a brief description of the constitutive models and the governing partial differential equations is provided. In Section 3 we give the variational formulation in the form of weak statement and minimization principle. We will also prove the well-posedness of the weak problem using the continuity and monotonicity of the associated abstract operator. In Section 4, we describe the divergence free finite element approximations and derive *a priori* finite element error estimates. In Section 5, we consider *a posteriori* error estimates for the finite element approximation. In Section 6 we examine the convergence of the method of successive approximations for solution of the non-linear equations. Finally, we briefly describe some theoretical results for non-stationary flows.

2. GENERALIZED NEWTONIAN FLUIDS

2.1. Fluid models

Let $\mathbf{u}$ be the fluid velocity and denote the rate of deformation tensor denoted by $\mathbf{D}$ and the shear rate by $s(\mathbf{u})$, i.e.,

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad s = s(\mathbf{u}) = \sqrt{2D_{ij}(\mathbf{u})D_{ij}(\mathbf{u})} \tag{1}$$

The class of generalized Newtonian fluids to be considered is characterized by the constitutive relation

$$\eta = \eta(s(\mathbf{u})) \tag{2}$$

and the requirement that the viscosity η and the derivative of $\eta(s)s$ are uniformly bounded above and below:

$$0 < m_0 \leqslant \eta(s) \leqslant M_0 < \infty \tag{3}$$

$$0 < m_1 \leqslant (\eta(s)s)' \leqslant M_1 < \infty \tag{4}$$

Typically, the viscosity is also a decreasing function.

Models of the generalized Newtonian fluids that satisfy these properties include:

1. Extended Williamson fluid

$$\eta(s) = \eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + (\lambda s)^{2-r}} \tag{5}$$

with viscosity parameters η_0, η_∞ and parameter λ satisfying $\eta_0 \geqslant \eta_\infty \geqslant 0$, $\lambda > 0$ and $1 \leqslant r \leqslant 2$. When $r = 2$, we recover the usual Newtonian fluid. For $r = 1$, we have the classical Williamson fluid model. For $r = \frac{4}{3}$, the fluid is sometimes referred to as Cross fluid.

2. Carreau fluid

$$\eta(s) = \eta_\infty + \frac{(\eta_0 - \eta_\infty)}{(1 + (\lambda s)^2)^{(2-r)/2}} \tag{6}$$

with parameters η_0, η_∞ and η satisfying $\eta_0 > \eta_\infty > 0$, $\lambda > 0$ and $r \geqslant 1$.

3. Eyring fluid

$$\eta(s) = \eta_\infty + (\eta_0 - \eta_\infty) \frac{\sinh^{-1} \lambda s}{\lambda s} \tag{7}$$

with parameters η_0, η_∞ and λ satisfying $\eta_0 > \eta_\infty > 0$, $\lambda > 0$.

4. Oldroyd model

$$\eta(s) = \eta_0 \frac{1 + (\lambda_1 s)^2}{1 + (\lambda_2 s)^2} \tag{8}$$

with positive parameters η_0, λ_1 and λ_2.

5. Generalized Oldroyd-B model with zero relaxation and retardation time

$$\eta(s) = \eta_\infty + (\eta_0 - \eta_\infty) \frac{1 + \log \lambda s}{1 + \lambda s} \tag{9}$$

with parameters η_0, η_∞ and λ satisfying $\eta_0 > \eta_\infty > 0$, $\lambda > 0$.

For many concentrated polymer solutions and melts, good fits can be obtained for $\eta_\infty = 0$. However, many theoretical results on stability and error estimates are meaningful only when η_∞ is positive. In this paper we consider the nondegenerate case $\eta_\infty > 0$ which would ensure (3) and (4) hold.

For example, from the definition (5) of the Williamson fluid viscosity η, it is easy to check that

$$\eta_\infty \leqslant \eta(s) \leqslant \eta_0 \quad \text{for all } s \geqslant 0 \tag{10}$$

$$(\eta(s)s)' = \eta_\infty + (\eta_0 - \eta_\infty) \frac{1 + (r-1)(\lambda s)^{2-r}}{(1 + (\lambda s)^{2-r})^2} \tag{11}$$

and so the boundedness conditions (3), (4) are satisfied. Similarly, one may check that other (nondegenerate) generalized Newtonian fluid models also satisfy these conditions.

2.2. Governing equations

Consider slow viscous flow in a bounded, simply connected domain $\Omega \subset \mathbb{R}^2$ with Lipschitz continuous boundary $\partial\Omega$. For non-stationary flows, the momentum equation is

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}\sigma_{ij} + f_i = 0 \quad \text{in } \Omega \times (0, T] \tag{12}$$

and, for stationary slow flow, we have

$$\frac{\partial}{\partial x_j}\sigma_{ij} + f_i = 0 \quad \text{in } \Omega \tag{13}$$

where $\mathbf{f}$ is the body force and $\boldsymbol{\sigma}$ is the stress tensor which is related to the pressure p, fluid velocity $\mathbf{u}$ and the viscosity η by

$$\sigma_{ij} = -p\delta_{ij} + 2\eta(s(\mathbf{u}))D_{ij}(\mathbf{u}) \tag{14}$$

Conservation of mass implies the incompressibility condition

$$\operatorname{div} \mathbf{u} = \frac{\partial u_i}{\partial x_i} = 0 \quad \text{in } \Omega \tag{15}$$

Boundary conditions complete the problem specification for the stationary flow problem. For simplicity, we consider homogeneous Dirichlet boundary data

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \tag{16}$$

but the analysis may be applied to other types of standard boundary conditions in a similar manner. For the unsteady problem, we also need an initial condition

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x) \quad \text{in } \Omega \tag{17}$$

for some given function $\mathbf{u}_0$.

3. VARIATIONAL FORMULATION

For fixed $q \in [1, 2]$ and integer k, let $V_{k,q} = \{\mathbf{v} \in [W_0^{k,q}(\Omega)]^2; \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega\}$ be the Sobolev space associated with the seminorm $\|\mathbf{v}\|_{k,q} = (\int_\Omega s(\mathbf{v})^q \, \mathrm{d}x)^{1/q}$. For $q > 1$, Korn's inequality implies here that the seminorm is equivalent to the usual Sobolev norm $\|\cdot\|_{k,q}$ on $V_{k,q}$. In what follows, when $k = 1$, $q = 2$ we drop the subscripts and set

$$V = \{\mathbf{v} \in (H_0^1(\Omega))^2; \ \operatorname{div} \mathbf{v} = 0 \ \text{in } \Omega\}$$

equipped with the corresponding norm $\|\cdot\|$. The dual space of V will be denoted by $(V^*, \|\cdot\|^*)$ and the corresponding duality pairing as $(\cdot, \cdot)$.

The variational principle corresponding to the equations (13–16) with the viscosity specified by (5–9) is

$$\min\{J(\mathbf{v});\ \ \mathbf{v}\in V\} \tag{18}$$

where

$$J(\mathbf{v}) = \int_\Omega \int_0^{s(\mathbf{v})} 2\eta(t)t\,\mathrm{d}t - \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \tag{19}$$

and the corresponding weak statement is given by

$$(J'(\mathbf{u}),\mathbf{v}) = 0 \quad \text{for all } \mathbf{v}\in V \tag{20}$$

where J' is the Gateaux derivative of J with

$$J'(\mathbf{u}) = A\mathbf{u} - \mathbf{f}$$

for

$$(A\mathbf{u},\mathbf{v}) = \int_\Omega 2\eta(s(\mathbf{u}))D_{ij}(\mathbf{u})D_{ij}(\mathbf{v})\,\mathrm{d}x$$

and

$$(\mathbf{f},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x$$

As we shall see below, existence and uniqueness are straight forward consequences of the strict monotonicity and continuity property of the operator A over the space V.

3.1. Well-posedness

From the boundedness properties (3), (4), one can show that the operator A is strongly monotone and continuous. The results are summarized below with the proofs omitted. Details are available in Reference [3].

Proposition 1
The operator A is Lipschitz continuous on V: there exists $M>0$ such that

$$\|A\mathbf{u} - A\mathbf{v}\|^* \leqslant M\ \|\mathbf{u}-\mathbf{v}\| \quad \text{for all } \mathbf{u},\mathbf{v}\in V \tag{21}$$

Proposition 2
The operator A is strongly monotone over V; i.e. for some $m>0$,

$$(A\mathbf{u} - A\mathbf{v},\mathbf{u}-\mathbf{v}) \geqslant m\ \|\mathbf{u}-\mathbf{v}\|^2 \quad \text{for all } \mathbf{u},\mathbf{v}\in V \tag{22}$$

As A is continuous over V, and A is strongly monotone, J is strictly coercive. Consequently, the function $J(\cdot)$ is convex, lower semi-continuous and coercive over V. Hence existence and uniqueness follows easily from standard theories on monotone operators and convex functionals. Furthermore, if $\mathbf{u_1},\mathbf{u_2}$ are solutions of (20) corresponding to data $\mathbf{f}=\mathbf{f_1},\mathbf{f_2}$, respectively, it is easy to show that

$$m\|\mathbf{u_1}-\mathbf{u_2}\|^2 \leqslant (A\mathbf{u_1} - A\mathbf{u_2},\mathbf{u_1}-\mathbf{u_2}) \leqslant M\ \|\mathbf{f_1}-\mathbf{f_2}\|^*\ \ \|\mathbf{u_1}-\mathbf{u_2}\|$$

Thus the solution to the variational problem (18) has continuous dependence on data. Hence (18) is a well-posed problem.

4. A CLASS OF DIVERGENCE FREE ELEMENTS

Let Ω be a simply connected, bounded domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. Consider the linear operator $\mathbf{curl}: H^1(\Omega) \to (L^2(\Omega))^2$ defined by

$$\mathbf{curl}\, w = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x} \right)$$

Let

$$H_0^2(\Omega) = \left\{ w \in H^2(\Omega);\; w = \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

where $\mathbf{n} = (n_1, n_2)$ is the unit outward normal.

As the operator *curl* is a bijection from $H_0^2(\Omega)$ onto V (see Reference [8]) we have

$$V = \mathbf{curl}\ H_0^2(\Omega) = \{\mathbf{v};\ \mathbf{v} = \mathbf{curl}\ w \text{ for some } w \in H_0^2(\Omega)\}$$

and so a divergence free finite element subspace V_h may be constructed by first constructing an arbitrary finite element subspace W_h in $H_0^2(\Omega)$ and then setting

$$V_h = \mathbf{curl}\ W_h = \{\mathbf{v}_h;\ \mathbf{v}_h = \mathbf{curl}\ w_h \text{ for some } w_h \in W_h(\Omega)\}$$

which then implies $V_h \subset V$, i.e. V_h consists of finite element functions that are divergence free.

The above idea is the basis of the construction of divergence free curl(PSH) elements using PSH elements. PSH finite element basis functions are continuously differentiable, piecewise quadratic polynomials defined on a macro triangle consisting of twelve subtriangles. Each element K has twelve degrees of freedom defined by nodal and derivative values at the vertices and normal derivative values defined at the midpoint of each side of the triangle (see Figure 1). A simpler composite Powell–Sabin 6-split element can also be constructed.

The Powell–Sabin elements were initially devised for use in computer graphics, especially contour plotting. Heindl developed the PSH element independently and studied its approximation properties. Křížek and Liu [8] were the first to consider using the PSH element to generate divergence free elements for use in Navier–Stokes calculation.

The curl(PSH) basis functions are continuous and piecewise linear over each subtriangle $T \in K$. The basis functions are listed in the appendix.

Let $\mathcal{T}_h$ be a triangulation based on PSH elements K. We shall assume that the triangulation $\mathcal{T}_h$ is quasi-regular in the sense that for all $T \in K$ and for all $K \in \mathcal{T}_h$, there exists constants C_1, C_2 such that

$$C_1 h^2 \leqslant \text{meas}(T) \leqslant \text{meas}(K) \leqslant C_2 h^2 \tag{23}$$

The curl(PSH) finite element approximation for (20) is given by:

Find $\mathbf{u}_h \in V_h$ such that

$$(A\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in V_h \tag{24}$$

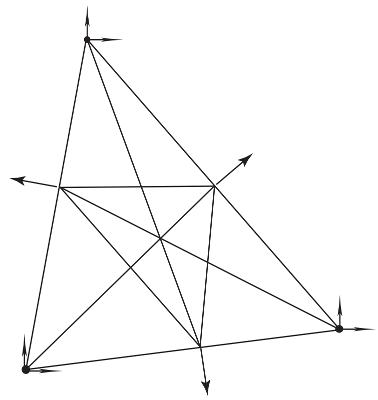

Figure 1. Powell–Sabin–Heindl element.

The minimization problem corresponding to the above formulation is to find $\mathbf{u}_h \in V_h$ such that

$$J(\mathbf{u}_h) = \min\{J(\mathbf{v}_h);\ \mathbf{v}_h \in V_h\} \tag{25}$$

From the continuity property (Proposition 1) and the monotonicity property (Proposition 2), we have the following:

Proposition 3
The solutions $\mathbf{u}$ and $\mathbf{u}_h$ are uniformly bounded above.

Proof
Since $A\mathbf{0} = \mathbf{0}$ and using Propositions 1 and 2, we have

$$m\,\|\mathbf{u}_h\|^2 \leqslant (A\mathbf{u}_h, \mathbf{u}_h) = (\mathbf{f}, \mathbf{u}_h) \leqslant \|\mathbf{f}\|^{*}\mathbf{u}_h\|$$

and thus

$$\|\mathbf{u}_h\| \leqslant C$$

with the constant C independent of h. A similar result holds for $\mathbf{u}$.

5. ENERGY AND L^q ESTIMATE

In this section we establish error estimates of the finite element approximations in the energy norm and in the L^q norm, with $1 < q < 2$. First we establish an abstract error estimate and then an energy estimate for the curl(PSH) finite element approximation. We then show how, under additional assumption on the viscosity, L^q error estimates may be established.

5.1. Abstract error estimate

Theorem 1
The finite element approximation $\mathbf{u}_h$ of (24) to the solution $\mathbf{u}$ of (20) *satisfies the following error bound*:

$$\|\mathbf{u}-\mathbf{u}_h\| \leqslant C \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u}-\mathbf{v}_h\| \tag{26}$$

Proof
From Propositions 1 and 2, for arbitrary $\mathbf{v}_h \in V_h$, using (24), we have

$$\begin{aligned} m\|\mathbf{u}-\mathbf{u}_h\|^2 &\leqslant (A\mathbf{u}-A\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) \\ &= (A\mathbf{u}-A\mathbf{u}_h, \mathbf{u}-\mathbf{v}_h) \\ &\leqslant \|A\mathbf{u}-A\mathbf{u}_h\|^* \, \|\mathbf{u}-\mathbf{v}_h\| \\ &\leqslant M\|\mathbf{u}-\mathbf{u}_h\| \; \|\mathbf{u}-\mathbf{v}_h\| \end{aligned}$$

and so the abstract estimate follows easily.

Theorem 2
The finite element solutions $\mathbf{u}_h$ of (24) converge to the solution $\mathbf{u}$ of (20).

Proof
As $h\to 0$, the finite element space $V_h \to V$, so Theorem 1 implies that $\mathbf{u}_h$ converges to $\mathbf{u}$ in the 'energy' norm as $h\to 0$.

5.2. Finite element error estimate

Let $\{W_h\}_{h\to 0}$ be a family of finite element subspaces of $H_0^2(\Omega)$ such that the W_h-interpolant $\Pi_h\psi$ of $\psi \in H_0^2(\Omega) \cap H^{k+2}(\Omega)$ possesses the approximation property

$$\|\psi - \Pi_h \psi\|_{H^{1,2}} \leqslant Ch^k \, |\psi|_{H^{k+2,2}} \tag{27}$$

Here C denotes a positive constant independent of h, and k is a given integer. For any $\mathbf{v}\in V\cap [H^{k+1}]^2$, there exists $\psi \in H_0^2(\Omega)$ such that $\mathbf{v}=\mathbf{curl}\,\psi$, since curl is a bijection. An interpolant $\Pi_h\mathbf{v}\in V_h$ of $\mathbf{v}$ may be constructed by setting

$$\Pi_h\mathbf{v} = \mathrm{curl}(\Pi_h\psi)$$

The interpolant $\Pi_h\mathbf{v}$ has the following approximation property (e.g. see Reference [8] for the 12-split case):

Theorem 3
If (27) holds, then for all $V\cap[H^{k+1}(\Omega)]^2$,

$$\|\mathbf{v}-\Pi_h\mathbf{v}\|_{1,2} \leqslant Ch^k |\mathbf{v}|_{k+1,2} \tag{28}$$

In particular, for the curl(PSH) elements considered here, $\Pi_h\mathbf{v}$ is piecewise linear and $k=1$. Hence from the abstract estimate (26) in Theorem 1, we have the following result.

Theorem 4
Let $\mathbf{u}$ and $\mathbf{u}_h$ be the solutions of (20) and (24) respectively. If $\mathbf{u} \in V \cap [H^{k+1}]^2$ then

$$\|\mathbf{u} - \mathbf{u}_h\|_1 \leqslant Ch|\mathbf{u}|_{k+1}$$

Proof
From Theorem 1,

$$\|\mathbf{u} - \mathbf{u}_h\| \leqslant C\|\mathbf{u} - \Pi_h \mathbf{u}\|$$

and applying the approximating property (28) of the interpolant Π_h, we have the desired result.

5.3. L^q-error estimates

From a theoretical viewpoint, an additional benefit of utilizing divergence free elements is that one may also derive L^q-error estimates for $1 < q < 2$. In this subsection, we show that if the viscosity function satisfies the property that there exists a positive constant C such that

$$|\eta'(s)| + |s\eta''(s) - \eta'(s)| \leqslant C \quad \text{for all } s \geqslant 0 \tag{29}$$

then it is possible to derive L^q-error estimates for the finite element approximations.

For linear problems, L^2-error estimates are obtained with the aid of Nitsche's duality argument which exploits the adjoint operator associated with the original problem. For non-linear operators, it is of course not possible to define an adjoint operator directly. However, if the operator A associated with a non-linear problem is differentiable, then, for each u in $V \subset \mathrm{dom}(A)$, the operator $A'(u)$ is a linear operator from V to V' and we may then construct an adjoint $(A'(u))^*$ with

$$(A'(\mathbf{u})\mathbf{v}, \mathbf{w}) = ((A'(\mathbf{u}))^* \mathbf{w}, \mathbf{v}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in V \tag{30}$$

For the generalized Newtonian flow problems considered here, let

$$f_{ijk\ell}(s(\mathbf{u})) = \eta(s(\mathbf{u}))\delta_{ik}\delta_{j\ell} + \frac{\eta'(s(\mathbf{u}))}{s(\mathbf{u})} D_{ij}(\mathbf{u}) D_{k\ell}(\mathbf{u}) \tag{31}$$

then

$$(A'(\mathbf{u})\mathbf{v}, \mathbf{w}) = \int_\Omega f_{ijk\ell}(s(\mathbf{u})) D_{ij}(\mathbf{v}) D_{k\ell}(\mathbf{w})\, \mathrm{d}x \tag{32}$$

that is,

$$\begin{aligned}(A'(\mathbf{u})\mathbf{v}, \mathbf{w}) = \int_\Omega & \eta(s(\mathbf{u}))\mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w}) \\ & + \frac{\eta'(s(\mathbf{u}))}{s(\mathbf{u})} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v})\, \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{w})\, \mathrm{d}x \end{aligned} \tag{33}$$

For each $\mathbf{v}, \mathbf{w} \in V$, we may also define an operator $A_h : V \to V^*$ by

$$(A_h \mathbf{v}, \mathbf{w}) = \int_\Omega a^h_{ijk\ell} D_{ij}(\mathbf{v}) D_{k\ell}(\mathbf{w})\, \mathrm{d}x \tag{34}$$

where

$$a^h_{ijk\ell} = \int_0^1 f_{ijk\ell}(\theta s(\mathbf{u}) + (1-\theta)s(\mathbf{u}_h))\,\mathrm{d}s \tag{35}$$

where $\mathbf{u}$ and $\mathbf{u}_h$ are the weak solution and the finite element approximations respectively. Using the mean value theorem and the orthogonal relation

$$(A\mathbf{u} - A\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \text{for all } \mathbf{v}_h \in V_h \tag{36}$$

we see that

$$(A_h(\mathbf{u} - \mathbf{u}_h), \mathbf{v}_h) = 0 \quad \text{for all } \mathbf{v}_h \in V_h \tag{37}$$

Note that for linear problems, $A = A'(\mathbf{v}) = A_h$. With these operators defined, we now proceed to prove the $\mathbf{L}^q$-error estimates. Let $1/p + 1/q = 1$.

Theorem 5
For problem (20), if $\mathbf{u} \in [W^{2,p}(\Omega)]^2$ and (29) holds, then the finite element solution to (24) satisfies

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,q} \leqslant C\,\|\mathbf{u} - \mathbf{u}_h\|^2 + Ch\|\mathbf{u} - \mathbf{u}_h\| \tag{38}$$

where $\mathbf{w}$ is the solution to the adjoint problem (40) below.

Proof
Let $\mathbf{g} \in [L^p(\Omega)]^2$ be given. Proceeding as in the standard Nitsche approach, we seek to establish the bound

$$\left| \int_\Omega \mathbf{g} \cdot (\mathbf{u} - \mathbf{u}_h)\,\mathrm{d}x \right| \leqslant Ch\|\mathbf{g}\|_{0,p} \tag{39}$$

where C is independent of h.

First, consider the adjoint problem:

$$\text{Find } \mathbf{w} \in V\text{: } ((A'(\mathbf{u}))^* \mathbf{w}, \mathbf{v}) = (\mathbf{g}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V \tag{40}$$

From (30), Equation (40) may be rewritten as

$$(A'(\mathbf{u})\mathbf{v}, \mathbf{w}) = \int_\Omega f_{ijk\ell} D_{ij}(\mathbf{v}) D_{k\ell}(\mathbf{w})\,\mathrm{d}x = (\mathbf{g}, \mathbf{v}) \tag{41}$$

Now, suppressing dependence on $\mathbf{u}$,

$$\begin{aligned}\frac{\partial}{\partial x_m} f_{ijk\ell}(x) = {} & \frac{\eta'(s)}{s}\,(D_{i'j',m} D_{i'j'} \delta_{ik}\delta_{j\ell} + D_{ij,m} D_{k\ell} + D_{ij} D_{k\ell,m}) \\ & + \left(\frac{\eta'(s)}{s}\right)' \frac{1}{s}\,(D_{i'j',m} D_{i'j'} D_{ij} D_{k\ell})\end{aligned} \tag{42}$$

so

$$\left|\frac{\partial}{\partial x_m} f_{ijk\ell}(x)\right| \leqslant C|\nabla^2 \mathbf{u}| \ (|\eta'(s)| + |s\eta''(s) - \eta'(s)|) \tag{43}$$

where $|\nabla^2 \mathbf{u}|$ denote some matrix norm of the Hessian matrix of $\mathbf{u}$ at some fixed $x \in \Omega$.

Consequently, if $\mathbf{u} \in [W^{2,p}(\Omega)]^2$ and (29) holds, then $f_{ijk\ell} \in W^{1,p}(\Omega)$. The standard regularity result [9] may now be invoked to yield the well-posedness of (40) and the inequality

$$\|\mathbf{w}\|_{2,p} \leqslant C \ \|\mathbf{g}\|_{0,\,p} \tag{44}$$

Letting $\mathbf{e}_h = \mathbf{u} - \mathbf{u}_h$ in (41) and using the triangle inequality we get

$$\left|\int_\Omega \mathbf{g} \cdot (\mathbf{u} - \mathbf{u}_h)\,\mathrm{d}x\right| \leqslant |(A'(\mathbf{u})(\mathbf{e}_h) - A_h(\mathbf{e}_h), \mathbf{w})| + |(A_h(\mathbf{e}_h), \mathbf{w})| \tag{45}$$

For the second term, we have

$$|(A_h(\mathbf{u} - \mathbf{u}_h), \mathbf{w})| = |(A_h(\mathbf{u} - \mathbf{u}_h), \mathbf{w} - \mathbf{w}_I)| \tag{46}$$

where $\mathbf{w}_I$ is an arbitrary element of V_h. So

$$|(A_h \mathbf{e}_h, \mathbf{w})| \leqslant C \ \|\mathbf{e}_h\| \ \|\mathbf{w} - \mathbf{w}_I\|_{0,\,p} \tag{47}$$

since $L^2 \hookrightarrow L^q$ and

$$|a^h_{ijk\ell}| \leqslant \int_0^1 (\eta(t)t)'|_{t=\theta s(\mathbf{u})+(1-\theta)s(\mathbf{u}_h)}\,\mathrm{d}\theta \leqslant C \tag{48}$$

Hence from approximation theory,

$$\inf\{\|\mathbf{w} - \mathbf{w}_I\|, \ \mathbf{w}_I \in V_h\} \leqslant C\,h\,\|\mathbf{w}\|_{2,p}$$

and (44), we have

$$|(A_h \mathbf{e}_h, \mathbf{w})| \leqslant C\,h\,\|\mathbf{e}_h\| \ \|\mathbf{g}\|_{0,\,p} \tag{49}$$

For the first term, since the derivatives of $f_{ijk\ell}$ are uniformly bounded, and the imbedding $W^{2,s}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ holds for $\Omega \subset \mathbb{R}^2$ and $s > 2$,

$$\begin{aligned}
&|(A'(\mathbf{u})(\mathbf{e}_h) - A_h(\mathbf{e}_h), \mathbf{w})| \\
&\quad \leqslant \int_0^1 \int_\Omega |f_{,ijk\ell}(\mathbf{u}) - f_{,ijk\ell}(\theta\mathbf{u} + (1-\theta)\mathbf{u}_h)|\, s(\mathbf{e}_h)\, s(\mathbf{w})\,\mathrm{d}x\,\mathrm{d}\theta \\
&\quad \leqslant C\|\mathbf{e}_h\|^2 \ \|\mathbf{w}\|_{1,\infty} \\
&\quad \leqslant C\|\mathbf{u} - \mathbf{u}_h\|^2 \ \|\mathbf{w}\|_{2,\,p}
\end{aligned} \tag{50}$$

So from (45), (49), (50), (44) we have

$$\int_\Omega \mathbf{g} \cdot (\mathbf{e}_h)\,\mathrm{d}x = (A'(\mathbf{u})\mathbf{e}_h - A_h\mathbf{e}_h, \mathbf{w}) + (A_h\mathbf{e}_h, \mathbf{w}) \tag{51}$$

and so

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,q} \leqslant C\,\|\mathbf{u}-\mathbf{u}_h\|^2 + C\,h\,\|\mathbf{u}-\mathbf{u}_h\| \tag{52}$$

which completes the proof.

For curl(PSH) finite elements $\|\mathbf{u}-\mathbf{u}_h\| = \mathcal{O}(h)$ and $\|\mathbf{w}-\mathbf{w}_h\| = \mathcal{O}(h)$, so we have the following:

Theorem 6
Under the conditions of Theorem 5, the curl(PSH) finite element approximations have the optimal order of convergence in the L^q-norm, i.e.

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,q} = \mathcal{O}(h^2)$$

5.3.1. Some specific fluid models. The above estimate was established under the assumption that the fluid model satisfies condition (29). We now show that it is satisfied by the fluid models listed in Section 2.1 except for the Williamson fluid with $1<r<2$ and the generalized Oldroyd-B model. For simplicity, let $\eta_0 - \eta_\infty = 1$. For the classical Williamson fluid (5),

$$|\eta'(s)| = \frac{\lambda}{(1+\lambda s)^2} \leqslant \lambda \tag{53}$$

$$|s\eta''(s) - \eta'(s)| = \lambda \frac{(3\lambda s + 1)}{(1+\lambda s)^3} \leqslant 4\lambda \tag{54}$$

and, for the Carreau fluids (6),

$$|\eta'(s)| = (2-r)\,\frac{\lambda^2 s}{(1+\lambda^2 s^2)^{(4-r)/2}} \leqslant (2-r)\lambda \tag{55}$$

$$\begin{aligned} |s\eta''(s) - \eta'(s)| &= (2-r)(4-r)\frac{\lambda^4 s^3}{(1+\lambda^2 s^2)^{3-(r/2)}} \\ &\leqslant (2-r)(4-r)\lambda \end{aligned} \tag{56}$$

For the Oldroyd model (8),

$$|\eta'(s)| = 2\,\frac{\eta_0 s |\lambda_1^2 - \lambda_2^2|}{(1+\lambda_2^2 s^2)^2} \leqslant \frac{2}{\lambda_2}|\lambda_1^2 - \lambda_2^2| \tag{57}$$

$$|s\eta''(s) - \eta'(s)| = 8\,\frac{\lambda_2^2 s^3 |\lambda_1^2 - \lambda_2^2|}{(1+\lambda_2^2 s^2)^3} \leqslant \frac{8}{\lambda_2}|\lambda_1^2 - \lambda_2^2| \tag{58}$$

For the Eyring fluid (7), the verification is slightly more involved.
To show the derivative

$$|\eta'(s)| = \left| \frac{\sinh^{-1}(\lambda s)}{\lambda s^2} - \frac{1}{s\sqrt{1+\lambda^2 s^2}} \right| \tag{59}$$

is bounded, we first consider the function

$$f(s) = \lambda^2 s^2 + \frac{\lambda s}{\sqrt{1+\lambda^2 s^2}} - \sinh^{-1} \lambda s \tag{60}$$

Clearly, $f(0)=0$. Since

$$f'(s) = 2\,\lambda^2 s - \frac{s^2 \lambda^3}{(1+\lambda^2 s^2)^{3/2}} \tag{61}$$

and

$$\frac{\lambda^3 s^2}{(1+\lambda^2 s^2)^{3/2}} = \lambda^2 s \, \frac{\lambda s^2}{(1+\lambda^2 s^2)^{3/2}} \leqslant \lambda^2 s$$

so $f'(s) \geqslant \lambda^2 s \geqslant 0$ and hence $f(s) \geqslant 0$ for all $s \geqslant 0$.

Furthermore, the function

$$g(s) = \sinh^{-1} \lambda s - \frac{\lambda s}{(1+\lambda^2 s^2)^{1/2}}$$

is also an non-negative function since $g(0)=0$ and

$$g'(s) = \frac{\lambda^3 s^2}{(1+\lambda^2 s^2)^{3/2}} \geqslant 0$$

Consequently, we have

$$|\eta'(s)| = \left| \frac{\sinh^{-1}(\lambda s)}{\lambda s^2} - \frac{1}{s\sqrt{1+\lambda^2 s^2}} \right| \leqslant \lambda \tag{62}$$

Furthermore, as

$$|s\eta''(s)| = \left| -\frac{\lambda^2 s}{(1+\lambda^2 s^2)^{3/2}} - 2\eta'(s) \right| \leqslant \lambda + 2|\eta'(s)| \leqslant 3\lambda \tag{63}$$

and hence the inequality (29) is satisfied.

6. *A POSTERIORI* ERROR ESTIMATES

Let us now consider the derivation of *a posteriori* upper and lower bounds for the finite element approximation error $\|\mathbf{u} - \mathbf{u}_h\|$. Such estimates are particularly relevant for analysis of the reliability of computed solutions and for adaptive mesh calculations.

For each PSH element $K \in \mathcal{T}_h$, let T denote one of the subtriangles over which a given finite element function $\mathbf{v}_h$ is linear, ∂T its boundary and $e \in \partial T$ be one of its *interior* edges. We use $[\tau_{ij} n_j]_e$ to denote the jump of the normal component of the tensor τ across e.

An upper bound for $\|\mathbf{u} - \mathbf{u}_h\|$ may be obtained using the idea of Durán and Padra [10]. The result stated in that paper concerns an error estimator for the p-Laplacian problem using linear Crouzeix–Raviart elements. Although there is a basic deficiency with that result in that some, usually uncomputable, constants arise in the error estimator for the p-Laplacian problem, there is no similar difficulty in our situation.

More specifically, for any $\mathbf{v}\in[H^1(\Omega)]^2$, let $D^h(\mathbf{v})$ be an L^2-tensor defined by

$$D^h_{ij}(\mathbf{v})|_T = D_{ij}(\mathbf{v}|_T) \tag{64}$$

Set

$$s^h(\mathbf{v}) = \sqrt{2D^h(\mathbf{v}):D^h(\mathbf{v})}$$

For given finite element approximation $\mathbf{u}_h\in V_h$ of the solution $\mathbf{u}$ of (20), we define the edgewise constant vector

$$\mathbf{J}_{e,n} = \begin{cases} [\eta(s(\mathbf{u}_h))D_{ij}(\mathbf{u}_h)n_j]_e & \text{if } e\subset\partial T,\ e\not\subset\partial\Omega \\ \mathbf{0} & \text{if } e\subset\partial\Omega \end{cases} \tag{65}$$

Finally, for fixed $\mathbf{u}_h\in V_h$, let $\mathbf{U}\in V$ be defined by

$$(A\mathbf{U},\mathbf{v}) = \int_\Omega 2\eta(s(\mathbf{u}_h))D_{ij}(\mathbf{u}_h)D_{ij}(\mathbf{v})\,\mathrm{d}x \quad \text{for all } \mathbf{v}\in V \tag{66}$$

The function $\mathbf{U}$ may be viewed as the extension of $\mathbf{u}_h$ on V.

The *a posteriori* error estimator is then constructed by establishing upper bounds for $\|\mathbf{u}-\mathbf{U}\|$ and $\|\mathbf{U}-\mathbf{u}_h\|$ and applying the triangle inequality.

For a bound on $\|\mathbf{u}-\mathbf{U}\|$, we have

Theorem 7

Let $\mathbf{u}\in V$ be the weak solution of (20) and let $\mathbf{u}_h\in V_h$ be its finite element approximation. Then there exists a constant C such that

$$\|\mathbf{u}-\mathbf{U}\| \leqslant C\left(\sum_{K\in\mathscr{T}_h}\sum_{T\subset K}\mathrm{meas}(\mathbf{T})\,\|\mathbf{f}\|^2_{L^2(T)}\right)^{1/2} \tag{67}$$

Proof

Let $\mathbf{v}\in V$. Since $D(\mathbf{u}_h)$ is piecewise constant over each curl(PSH) triangle element, from (20) and (24), we may integrate by parts to get

$$\begin{aligned}(A\mathbf{u}-A\mathbf{u}_h,\mathbf{v}) &= \sum_{K\in\mathscr{T}_h}\sum_{T\subset K}\int_T \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x - \int_e 2\eta(s(\mathbf{u}_h))D_{ij}(\mathbf{u}_h)n_j\,\mathbf{v}_i\,\mathrm{d}s \\ &= \sum_{K\in\mathscr{T}_h}\left(\sum_{T\subset K}\int_T \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x + \sum_{e\subset\partial T}\int_e \mathbf{J}_{e,n}\cdot\mathbf{v}\,\mathrm{d}s\right)\end{aligned} \tag{68}$$

Note that if $\mathbf{v}=\mathbf{v}_h\in V_h$, then

$$\sum_{K\in\mathscr{T}_h}\sum_{T\subset K}\int_T \mathbf{f}\cdot\mathbf{v}_h\,\mathrm{d}x + \sum_{e\subset\partial T}\int_e \mathbf{J}_{e,n}\cdot\mathbf{v}_h\,\mathrm{d}s = 0 \tag{69}$$

From Proposition 2, integrating by parts and recalling that $D_{ij}(\mathbf{u}_h)$ is piecewise constant,

$$
\begin{aligned}
m\|\mathbf{u}-\mathbf{u}_h\|^2 & \\
&\leqslant (A\mathbf{u}-A\mathbf{U},\mathbf{u}-\mathbf{u}_h) \\
&= \sum_{K\in\mathscr{T}_h}\left(\sum_{T\subset K}\int_T \mathbf{f}\cdot(\mathbf{u}-\mathbf{u})_{\mathbf{h}}\,\mathrm{d}x+\sum_{e\subset\partial T}\int_e \mathbf{J}_{e,n}\cdot(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}s\right)
\end{aligned}
\tag{70}
$$

If we let $\mathbf{v}_h$ in (69) be $\mathbf{e}_h^I$, the approximation of $\mathbf{e}\equiv\mathbf{u}-\mathbf{u}_h$ in V_h defined by

$$\int_e \mathbf{e}_h^I\,\mathrm{d}s=\int_e \mathbf{e}\,\mathrm{d}s \tag{71}$$

then from (69)–(71),

$$m\|\mathbf{e}\|^2\leqslant\sum_{K\in\mathscr{T}_h}\sum_{T\subset K}\int_T \mathbf{f}\cdot(\mathbf{e}-\mathbf{e}_h^I)\,\mathrm{d}x \tag{72}$$

Applying the Cauchy Schwarz inequality and the interpolation error bound

$$\|\mathbf{e}-\mathbf{e}_h^I\|_{0,2,T}\leqslant C\left(\sum_{T\subset K}\mathrm{meas}(T)\right)^{1/2}\|\mathbf{e}\| \tag{73}$$

we have (67) and thus the theorem is proved.

Next, for a bound on $\|\mathbf{U}-\mathbf{u}_h\|$, let E_I denote the set of interior edges of the PSH elements, and set

$$\mathbf{J}_{e,t}=\begin{cases}[\frac{\partial u_h}{\partial t}]_e & \text{if } e\in E_I\\ -2\frac{\partial u_h}{\partial t}|_e & \text{if } e\subset\partial\Omega\end{cases} \tag{74}$$

where $[\partial u_h/\partial t]_e$ denotes the jump across e of the tangential derivative.

Theorem 8

Let $\mathbf{u}\in\mathbf{V}$ be the weak solution of (20) and let $\mathbf{U}\in V$ be an extension of the finite element approximation $\mathbf{u}_h\in V_h$ as defined in (66), then there exists a (computable) constant C_N such that

$$\|\mathbf{U}-\mathbf{u}_h\|\leqslant C_N\left(\sum_{K\in\mathscr{T}_h}\sum_{T\in K}\sum_{e\in\partial T}\mathrm{meas}(e)^2|\mathbf{J}_{e,t}|^2\right)^{1/2}$$

Proof

Using Proposition 2 we have

$$\|D(\mathbf{U}-\mathbf{u}_h)\|^2_{L^2(\Omega)}\leqslant C\int_\Omega(\eta(s(\mathbf{u}_h))D(\mathbf{u}_h)-\eta(s(\mathbf{U}))D(\mathbf{u})):D(\mathbf{u}_h)\,\mathrm{d}x$$

From (66)

$$\mathrm{div}(\eta(s(\mathbf{U}))D(\mathbf{U})-\eta(s^h(\mathbf{u}_h))D^h(\mathbf{u}_h))=0$$

there exists $\phi \in H^1(\Omega)$ such that

$$\eta(s(\mathbf{U}))D(\mathbf{U}) - \eta(s^h(\mathbf{u}_h))D^h(\mathbf{u}_h) = \operatorname{curl}\phi \tag{75}$$

and thus from the orthogonality relation

$$\sum_{K\in\mathscr{T}_h}\sum_{T\subset K}\int_T D^h(e)\operatorname{curl}\mathbf{v}\,\mathrm{d}x = 0$$

for all piecewise linear and continuous $\mathbf{v}\in V_h$, we have

$$\|D^h(\mathbf{U}-\mathbf{u}_h)\|^2_{L^2(\Omega)} \tag{76}$$

$$\leqslant C\int_\Omega D^h(\mathbf{u}_h)\operatorname{curl}(\phi-\phi^I)\,\mathrm{d}x \tag{77}$$

$$= C\sum_{K\in\mathscr{T}_h}\sum_{T\in K}\sum_{e\in\partial T}\int_e J_{e,t}(\phi-\phi^I)\,\mathrm{d}x \tag{78}$$

$$\leqslant C\left(\sum_{K\in\mathscr{T}_h}\sum_{T\in K}\sum_{e\in\partial T}\operatorname{meas}(e)^2|\mathbf{J}_{e,t}|^2\right)^{1/2}|\phi|_{1,2} \tag{79}$$

From (75) and Proposition 1, we have

$$|\phi|_{1,2} \leqslant Cs^h(\mathbf{U}-\mathbf{u}_h) \tag{80}$$

and thus the theorem is proved.

We remark that the constants C in the above theorems are independent of the weak solution $\mathbf{u}$ and determinable using the parameters in (22), (73) and the geometric parameters of the triangulation.

It is also possible to compute a lower bound for the finite element error when two finite element approximations $\mathbf{u}_h\in V_h$ and $\mathbf{u}_{h'}\in V_{h'}$, with $h'<h$, to the weak solution $\mathbf{u}$ of (20) are available.

Since $V_h\subset V_{h'}\subset V$, from the minimizing property of the weak solution and the finite element solutions,

$$J(\mathbf{u}) \leqslant J(\mathbf{u}_{h'}) \leqslant J(\mathbf{u}_h)$$

and so

$$J(\mathbf{u}_h) - J(\mathbf{u}) \geqslant J(\mathbf{u}_{h'}) - J(\mathbf{u})$$

Also,

$$J(\mathbf{u}_h) - J(\mathbf{u}_{h'}) \geqslant 0 \geqslant J(\mathbf{u}) - J(\mathbf{u}_{h'})$$

As

$$(J'(\mathbf{u}_{h'}), \mathbf{u}_h - \mathbf{u}_{h'}) = 0$$

so using the monotonicity property (Proposition 2)

$$\begin{aligned}
J(\mathbf{u}_h) - J(\mathbf{u}_{h'}) &= \int_0^1 (J'(\mathbf{u}_h + \theta(\mathbf{u}_{h'} - \mathbf{u}_h)), \mathbf{u}_h - \mathbf{u}_{h'})\,\mathrm{d}\theta \\
&= \int_0^1 (J'(\mathbf{u}_h + \theta(\mathbf{u}_{h'} - \mathbf{u}_h)) - J'(\mathbf{u}_{h'}), \mathbf{u}_h - \mathbf{u}_{h'})\,\mathrm{d}\theta \\
&\geqslant \int_0^1 \left(\frac{m}{1-\theta}\right) \|\theta \mathbf{u}_{h'} + (1-\theta)\mathbf{u}_h - \mathbf{u}_{h'}\|^2 \mathrm{d}\theta \\
&\geqslant \frac{1}{2} m \ \|\mathbf{u}_h - \mathbf{u}_{h'}\|^2
\end{aligned}$$

Now

$$\begin{aligned}
J(\mathbf{u}_h) - J(\mathbf{u}) &= \int_0^1 (J'(\mathbf{u}_h + \theta(\mathbf{u} - \mathbf{u}_h)), \mathbf{u}_h - \mathbf{u})\,\mathrm{d}\theta \\
&= \int_0^1 (J'(\mathbf{u}_h + \theta(\mathbf{u} - \mathbf{u}_h)) - J'(\mathbf{u}), \mathbf{u}_h - \mathbf{u})\,\mathrm{d}\theta \\
&\leqslant \int_0^1 \|J'(\mathbf{u}_h + \theta(\mathbf{u} - \mathbf{u}_h)) - J'(\mathbf{u})\|^* \ \|\mathbf{u}_h - \mathbf{u}\|\,\mathrm{d}\theta \\
&\leqslant M \int_0^1 \|\mathbf{u}_h + \theta(\mathbf{u} - \mathbf{u}_h)\|\,\mathrm{d}\theta \cdot \|\mathbf{u}_h - \mathbf{u}\| \\
&= \frac{M}{2} \|\mathbf{u} - \mathbf{u}_h\|^2
\end{aligned}$$

Since

$$\begin{aligned}
\frac{1}{2} m \|\mathbf{u}_h - \mathbf{u}_{h'}\|^2 &\leqslant J(\mathbf{u}_h) - J(\mathbf{u}_{h'}) \\
&= J(\mathbf{u}_h) - J(\mathbf{u}) + J(\mathbf{u}) - J(\mathbf{u}_{h'}) \\
&\leqslant J(\mathbf{u}_h) - J(\mathbf{u}) \\
&\leqslant \frac{M}{2} \|\mathbf{u} - \mathbf{u}_h\|^2
\end{aligned} \tag{81}$$

so

$$\|\mathbf{u} - \mathbf{u}_h\| \geqslant \sqrt{\frac{m}{M}} \|\mathbf{u}_h - \mathbf{u}_{h'}\| \tag{82}$$

7. METHOD OF SUCCESSIVE APPROXIMATIONS

As the discretized problem is non-linear, a simple numerical algorithm to linearize the problem is the method of successive approximations, or sometimes known in the present context as the Kačanov method. With an initial guess $\mathbf{u}_h^{(0)} \in V_h$, the method of successive approximations generates a sequence of approximate solutions $\{\mathbf{u}_h^{(k)}\} \subset V_h$ by solving, for $k=0,1,2,\ldots,$

$$\int_\Omega 2\eta(s(\mathbf{u}_h^{(k)}))D_{ij}(\mathbf{u}_h^{(k+1)})D_{ij}(\mathbf{v}_h)\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{v}_h\,\mathrm{d}x \quad \text{for all } \mathbf{v}_h \in V_h \tag{83}$$

In Reference [11], this method applied to a quasi-Newtonian flow obeying the Carreau law was considered. It was shown that the Kačanov method generates a sequence of solutions $\{\mathbf{u}^{(k)}\}_{k=0}^\infty \subset V$ for the continuous problems, with $k=0,1,2,\ldots$:

$$\int_\Omega 2\eta(s(\mathbf{u}^{(k)}))D_{ij}(\mathbf{u}^{(k+1)})D_{ij}(\mathbf{v})\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \quad \text{for all } \mathbf{v} \in V \tag{84}$$

where $\eta(\cdot)$ is given by (6) and that the sequence converges to the weak solution $\mathbf{u}$ of (20) as $k\to\infty$.

The convergence result does not apply directly to the discretized problem in a mixed finite element method setting. However, in our context of divergence free elements, it is possible to prove convergence of the Kačanov method (83).

Let

$$B(\mathbf{v};\mathbf{w}_1,\mathbf{w}_2) = \int_\Omega 2\eta(s(\mathbf{v}))D_{ij}(\mathbf{w}_1)D_{ij}(\mathbf{w}_2)\,\mathrm{d}x$$

To further simply notation, we write, for fixed $\mathbf{u}_h^{(k)}$,

$$B_k(\mathbf{v}_h,\mathbf{w}_h) = B(\mathbf{u}_h^{(k)};\ \mathbf{v}_h,\mathbf{w}_h) \quad \text{for all } \mathbf{v}_h,\mathbf{w}_h \in V_h$$

Then (83) may be written as

$$B(\mathbf{u}_h^{(k)};\mathbf{u}_h^{(k+1)},\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in V_h \tag{85}$$

Theorem 9

Assuming that in addition to the bounds (3) and (4), the viscosity function $\eta(s)$ is a decreasing function. Then the Kačanov method (83) has a unique solution $\mathbf{u}_h^{(k+1)}$ for each $k\geqslant 1$, and the sequence $\{\mathbf{u}_h^{(k)}\}_{k=0}^\infty$ converges to the finite element solution $\mathbf{u}_h$ of (24) in V_h.

Proof

Following Reference [11], first note that the linear problem (83) corresponds to the minimization problem:

Find $\mathbf{u}_h^{(k+1)} \in V_h$ s.t.

$$J_k(\mathbf{u}_h^{(k+1)}) = \min\{J_k(\mathbf{v_h});\ \mathbf{v}_h \in V_h\} \tag{86}$$

where

$$J_k(\mathbf{v}_h) = \int_\Omega \frac{1}{2}\,\eta(s(\mathbf{u}_h^{(k)}))s^2(\mathbf{v}_h) - \mathbf{f}\cdot\mathbf{v}_h\,\mathrm{d}x \tag{87}$$

In view of bounds (3) and (4), existence and uniqueness follow from the strong convexity and continuity of $J_k(\cdot)$ over V_h.

Also, from the bound (3), we have

$$B_k(\mathbf{u}_h^{(k+1)}-\mathbf{u}_h^{(k)},\mathbf{u}_h^{(k+1)}-\mathbf{u}_h^{(k)})\geqslant m\|\mathbf{u}_h^{(k+1)}-\mathbf{u}_h^{(k)}\|^2 \tag{88}$$

for some $m>0$.

Next we show that the sequence $\{J(\mathbf{u}_h^{(k)})\}_{k=0}^{\infty}$ is decreasing. Here J is the functional defined in (19). An important immediate result is to show that

$$E(\mathbf{v})-E(\mathbf{u})\leqslant\frac{1}{2}\,[B(\mathbf{u};\mathbf{v},\mathbf{v})-B(\mathbf{u};\mathbf{u},\mathbf{u})] \tag{89}$$

where

$$E(\mathbf{v})=\int_\Omega\int_0^{s(\mathbf{v})}\eta(z)z\,\mathrm{d}z\,\mathrm{d}x$$

and

$$\begin{aligned}
&\frac{1}{2}\,[B(\mathbf{u};\mathbf{v},\mathbf{v})-B(\mathbf{u};\mathbf{u},\mathbf{u})]\\
&\quad=\frac{1}{2}\left[\int_\Omega 2\eta(s(\mathbf{u}))D_{ij}(\mathbf{v})D_{ij}(\mathbf{v})\,\mathrm{d}x-\int_\Omega 2\eta(s(\mathbf{u}))D_{ij}(\mathbf{v})D_{ij}(\mathbf{v})\,\mathrm{d}x\right]\\
&\quad=\int_\Omega \eta(s(\mathbf{u}))(s(\mathbf{v})^2-s(\mathbf{u})^2))\,\mathrm{d}x
\end{aligned}$$

Inequality (89) will hold if we can show that

$$\int_s^t \eta(z)z\,\mathrm{d}z\leqslant\frac{1}{2}\,\eta(s)(t^2-s^2)\quad\text{for all } t,s\geqslant 0$$

or, equivalently,

$$\int_s^t(\eta(z)-\eta(s))z\,\mathrm{d}z\leqslant 0\quad\text{for all } t,s\geqslant 0 \tag{90}$$

Let s be fixed and consider the function

$$g(t)=\int_s^t(\eta(z)-\eta(s))z\,\mathrm{d}z$$

For $t\geqslant s\geqslant 0$, since η is a decreasing function, $\eta(z)\leqslant\eta(s)$ for all $z\geqslant s$ and so $g(t)\leqslant 0$ for all $t\geqslant s\geqslant 0$.

If $s\geqslant t\geqslant 0$, we have

$$g(t)=\int_t^s(\eta(s)-\eta(z))z\,\mathrm{d}z$$

Using the fact that η is a decreasing function, $\eta(s)\leqslant\eta(z)$ for all $z\geqslant s$ and so again $g(t)\leqslant 0$ and thus (90) and (89) hold.

Consequently, from (88) and (89),

$$\begin{aligned} m\|\mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}\|^2 & \\ &\leqslant B_k(\mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}, \mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}) \\ &= B_k(\mathbf{u}_h^{(k)}, \mathbf{u}_h^{(k)}) - B_k(\mathbf{u}_h^{(k+1)}, \mathbf{u}_h^{(k+1)}) - 2B_k(\mathbf{u}_h^{(k+1)}, \mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}) \\ &= 2(E(\mathbf{u}_h^{(k)}) - E(\mathbf{u}_h^{(k+1)})) - 2(\mathbf{f}, \mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}) \\ &\leqslant 2(J(\mathbf{u}_h^{(k)}) - J(\mathbf{u}_h^{(k+1)})) \end{aligned} \tag{91}$$

Hence the sequence $\{J(\mathbf{u}_h^{(k)})\}_{k=0}^{\infty}$ is decreasing. From (25), $\{J(\mathbf{u}_h^{(k)})\}$ is bounded below by $J(\mathbf{u}_h)$ over V_h, hence

$$\lim_{k\to\infty} \|\mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}\| = 0 \tag{92}$$

Furthermore,

$$\begin{aligned} m\|\mathbf{u}_h^{(k)} - \mathbf{u}_h\|^2 &\leqslant (A\mathbf{u}_h^{(k)} - A\mathbf{u}_h, \mathbf{u}_h^{(k)} - \mathbf{u}_h) \\ &= B_k(\mathbf{u}_h^{(k)}, \mathbf{u}_h^{(k)} - \mathbf{u}_h) - (\mathbf{f}, \mathbf{u}_h^{(k)} - \mathbf{u}_h) \\ &= B_k(\mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}, \mathbf{u}_h^{(k)} - \mathbf{u}_h) \\ &\leqslant M\|\mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}\| \; \|\mathbf{u}_h^{(k)} - \mathbf{u}_h\| \end{aligned} \tag{93}$$

So

$$\lim_{k\to\infty} \|\mathbf{u}_h^{(k)} - \mathbf{u}_h\| \leqslant \lim_{k\to\infty} \frac{M}{m} \|\mathbf{u}_h^{(k)} - \mathbf{u}_h^{(k+1)}\| = 0 \tag{94}$$

It is worth mentioning that one can also develop *a posteriori* error estimates for the Kačanov iterates $\mathbf{u}_h^{(k)}$ by using a conjugate functional as outlined in Reference [11].

8. NON-STATIONARY FLOWS

The above analysis may be extended to non-stationary flows. The governing equations are given by the unsteady momentum equation (12) together with the constitutive equation (14), the incompressibility condition (15), the boundary condition (16) and the initial condition (17).

For $q, r \in [1, \infty]$ and integer m, let $\mathscr{W}$ be a subspace of $[W^{m,r}(\Omega)]^2$. We use $L^q(0, T; \mathscr{W})$ to denote the Banach space of L^q functions mapping $(0, T)$ into $\mathscr{W}$ with norm

$$\|\mathbf{v}\|_{(q;m,r)} = \|\mathbf{v}\|_{L^q(0,T;[W^{m,r}(\Omega)]^2)} = \left(\int_0^T \|\mathbf{v}(t)\|_{m,r}^q \, \mathrm{d}t \right)^{1/q}$$

for $1 \leqslant q < \infty$ and the usual sup norm definition for $q = \infty$. The spaces $H^k(0, T; \mathscr{W})$ and $C^k((0, T]; \mathscr{W})$ for integer $k \geqslant 0$ and related seminorms are similarly defined.

In what follows, we assume that $\mathbf{f}$ is Lipschitz continuous in time and $\mathbf{u}_0 \in [H^1(\Omega)]^2$.

For V defined as in Section 3, the weak solution $\mathbf{u} \in H^1(0, T; V)$ of the non-stationary flow problem is given by

$$\left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}\right) = (A\mathbf{u}, \mathbf{v}) - (\mathbf{f}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V \tag{95}$$

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x) \quad \text{for } x \in \Omega \tag{96}$$

The following stability result may be obtained using, e.g., the argument of Wei [12]:

Proposition 4

The unsteady weak problem (95) has a unique solution $\mathbf{u} \in C^1([0, t], V)$ such that

$$|\mathbf{u}_t|^2_{(2;0,2)} + |\mathbf{u}|^2_{L^\infty(0,T;V)} \leqslant C(|\mathbf{f}|^2_{(2;0,2)} + |\mathbf{u}_0|^2_1)$$

8.1. Semidiscrete approximation

Let V_h be the curl(PSH) finite element space defined in Section 4. The semidiscrete approximation to the weak problem (95)–(96) is

Find $\mathbf{u}_h \in H^1(0, T; V_h)$ such that for almost all $t \in [0, T]$,

$$\left(\frac{\partial}{\partial t}\mathbf{u}_h, \mathbf{v}_h\right) + (A\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in V_h \tag{97}$$

$$\mathbf{u}_h(x, 0) = \mathbf{u}_{0h}(x) \quad \text{for } x \in \Omega \tag{98}$$

where $\mathbf{u}_{0h} \in V_h$ is a projection of $\mathbf{u}_0$.

Proposition 5

The semidiscrete finite element approximation (97)–(98) to the unsteady weak problem (95)–(96) has a unique solution $\mathbf{u}_h \in C^1([0, t], V_h)$ such that

$$\|\mathbf{u}_{ht}\|^2_{(2;0,2)} + \|\mathbf{u}_h\|^2_{L^\infty(0,T;V)} \leqslant C(\|\mathbf{f}\|^2_{(2;0,2)} + \|\mathbf{u}_{0h}\|^2_1) \tag{99}$$

Theorem 10

With $\mathbf{u}_{0h}$ defined as the interpolation of $\mathbf{u}_0$, and assuming that $\mathbf{u} \in L^2(0, T; V) \cap L^2(0, T; H^2(\Omega))$, we have, for some constant $C > 0$,

$$\|\mathbf{u} - \mathbf{u}_h\|^2_{C([0,T];[L^2(\Omega)]^2)} + \|\mathbf{u} - \mathbf{u}_h\|^2_{L^2(0,T;V)} \leqslant Ch^2 \tag{100}$$

Proof

Let $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. From (95) and (97), we have

$$(\mathbf{e}_t, \mathbf{v}_h) + (A\mathbf{u} - A\mathbf{u}_h, \mathbf{v}_h) = 0 \quad \text{for all } \mathbf{v}_h \in V_h \tag{101}$$

Now

$$\int_0^t (\mathbf{e}_t, \mathbf{e})\,\mathrm{d}t = \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|\mathbf{e}(t)\|^2_{0,2}\right)\mathrm{d}t = \frac{1}{2}(\|\mathbf{e}(t)\|^2_{0,2} - \|\mathbf{u}_0 - \mathbf{u}_{0h}\|^2_{0,2}) \tag{102}$$

We get for $t \in (0,T]$, and for any $\mathbf{w}_h \in L^2(0,T;V_h)$,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}\|_{0,2}^2 + m\|\mathbf{e}\|^2 \\
&\quad \leqslant (\mathbf{e}_t, \mathbf{u}-\mathbf{u}_h) + (A\mathbf{u} - A\mathbf{u}_h, \mathbf{u}-\mathbf{u}_h) \\
&\quad = (\mathbf{e}_t, \mathbf{u}-\mathbf{w}_h) + (\mathbf{e}_t, \mathbf{w}_h - \mathbf{u}_h) \\
&\qquad + (A\mathbf{u} - A\mathbf{w}_h, \mathbf{u}-\mathbf{u}_h) + (A\mathbf{u} - A\mathbf{w}_h, \mathbf{w}_h - \mathbf{u}_h) \\
&\quad = (\mathbf{e}_t, \mathbf{u}-\mathbf{w}_h) + (A\mathbf{u} - A\mathbf{u}_h, \mathbf{u}-\mathbf{w}_h)
\end{aligned} \tag{103}$$

So by integrating in time, and recalling Propositions 1, 4 and 5, and applying the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\frac{1}{2}\|\mathbf{e}\|_{0,2}^2 + m\int_0^t \|\mathbf{e}\|^2\,\mathrm{d}t \\
&= \int_0^t (\mathbf{e}_t, \mathbf{u}-\mathbf{w}_h) + (A\mathbf{u} - A\mathbf{u}_h, \mathbf{u}-\mathbf{w}_h)\,\mathrm{d}t + \frac{1}{2}\|\mathbf{u}_0 - \mathbf{u}_{0h}\|_{0,2}^2 \\
&\leqslant \|\mathbf{e}_t\|_{(2;0,2)}\,\|\mathbf{u}-\mathbf{w}_h\|_{(2;0,2)} + \int_0^t \|A\mathbf{u} - A\mathbf{u}_h\|^*\|\mathbf{u}-\mathbf{w}_h\|\,\mathrm{d}t + \frac{1}{2}\|\mathbf{u}_0 - \mathbf{u}_{0h}\|_{0,2}^2 \\
&\leqslant C\ \|\mathbf{u}-\mathbf{w}_h\|_{(2;0,2)} + M\int_0^t \|\mathbf{e}\|\ \|\mathbf{u}-\mathbf{w}_h\|\,\mathrm{d}t + \frac{1}{2}\|\mathbf{u}_0 - \mathbf{u}_{0h}\|_{0,2}^2 \\
&\leqslant C\ \|\mathbf{u}-\mathbf{w}_h\|_{(2;0,2)} + \frac{m}{2}\|\mathbf{e}\|_{L^2(0,t;V)}^2 + \frac{2M^2}{m}\|\mathbf{u}-\mathbf{w}_h\|_{(2;1,2)}^2 \\
&\quad + \frac{1}{2}\|\mathbf{u}_0 - \mathbf{u}_{0h}\|_{0,2}^2
\end{aligned} \tag{104}$$

Note here that the constant C depends on $\mathbf{f}, \mathbf{u}_0$, and $\mathbf{u}_{0h}$. Combining (102), (104), and the interpolation error bound

$$\|\mathbf{u}_0 - \mathbf{u}_{0h}\|_{0,2} \leqslant C\|\mathbf{u}_0\|_{1,2}^2$$

we get

$$\|\mathbf{u} - \mathbf{u}_h(t)\|^2 + \|\mathbf{u}-\mathbf{u}_h\|_{L^2(0,t;V)}^2 \leqslant Ch^2 \tag{105}$$

The theorem follows by taking the supremum over $[0,T]$ in the above inequality.

8.2. Fully discrete approximation

A fully discrete approximation is obtained by applying the fully implicit method to (97). Let $\Delta t = T/N$, and $t_n = n\Delta t$ for $n = 0,\ldots,N$.

For $n = 1,\ldots,N$, find $\mathbf{u}_h^n \in V_h$ such that

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t}, \mathbf{v}_h\right) + (A\mathbf{u}_h^n, \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in V_h \tag{106}$$

and

$$\mathbf{u}_h^0 = \mathbf{u}_{0h} \quad \text{in } \Omega \tag{107}$$

with $\mathbf{f}^n$ denoting the value of $\mathbf{f}$ at $t = n\Delta t$.

Let

$$\mathbf{U}(t) = \mathbf{u}_h^n \frac{t - t_{n-1}}{\Delta t} + \mathbf{u}_h^{n-1} \frac{t_n - t}{\Delta t}, \quad t_{n-1} \leqslant t \leqslant t_n$$

$$\mathbf{U}_h^n(t) = \mathbf{U}(t_n) = \mathbf{u}_h^n, \quad t_{n-1} < t \leqslant t_n$$

To derive a bound for $\mathbf{u}(t) - \mathbf{U}(t)$, we first introduce an approximation $\mathbf{W}(t)$, which is piecewise linear in time, to the weak solution $\mathbf{u}(t)$.

For $n = 1, \ldots, N$ let $\mathbf{w}^n \in V$ be obtained by time discretized scheme

$$\left(\frac{\mathbf{w}^n - \mathbf{w}^{n-1}}{\Delta t}, \mathbf{v} \right) + (A\mathbf{w}^n, \mathbf{v}) = (\mathbf{f}^n, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V \tag{108}$$

and

$$\mathbf{w}^0 = \mathbf{u}_0 \quad \text{in } \Omega \tag{109}$$

We define

$$\mathbf{W}(t) = \mathbf{w}^n \frac{t - t_{n-1}}{\Delta t} + \mathbf{w}^{n-1} \frac{t_n - t}{\Delta t}, \quad t_{n-1} \leqslant t \leqslant t_n$$

With an argument analogous to that of Wei [12], one can show that the sequence $\mathbf{w}^n$, and consequently $\mathbf{W}(t)$ is well defined; the sequence $\{\mathbf{w}^n\}$ is uniformly bounded in the L^2 and the H^1 norms (in space); the function $\{\mathbf{W}(t)\}$ and its time derivative are, for fixed $t \in [0, T]$, uniformly bounded in the L^2 norm in space.

Furthermore, if we write $W_k(t)$ for the function $\mathbf{W}(t)$ corresponding to $\Delta t = T/k$, then the sequence $\{\mathbf{W}_k\}$ is a Cauchy sequence converging to the weak solution $\mathbf{u}(t)$ as $k \to \infty$, with

$$\|\mathbf{W}_m - \mathbf{W}_k\|_{0,2}^2 \leqslant C \left(\frac{1}{m} + \frac{1}{k} \right) \tag{110}$$

and consequently, for fixed $t \in [0, T]$,

$$\|\mathbf{u}(t) - \mathbf{W}(t)\|_{0,2}^2 \leqslant C \frac{1}{N} = C \ \Delta t \tag{111}$$

Let $W_h^n \in V_h$ be the projection of $\mathbf{w}^n$ onto V_h, i.e.

$$(AW_h^n - A\mathbf{u}^n, \mathbf{v}_h) = 0 \quad \text{for all } \mathbf{v}_h \in V_h \tag{112}$$

and so from (106), (108), and (112), and setting $\mathbf{v}_h = \mathbf{W}_h^n - \mathbf{w}_h^n$,

$$\left(\frac{\mathbf{w}^n - \mathbf{w}^{n-1}}{\Delta t} - \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t}, \mathbf{W}_h^n - \mathbf{u}_h^n \right) + (A\mathbf{W}_h^n - A\mathbf{u}_h^n, \mathbf{W}_h^n - \mathbf{u}_h^n) = 0 \tag{113}$$

Theorem 11

The fully discrete finite element approximation (106)–(107) to the unsteady weak problem (95)–(96) has a unique solution $\mathbf{u}_h^n \in V_h$ for each $n \geqslant 1$ and, with $\mathbf{u}_{0h}$ defined as in the previous theorem, and assuming that $\mathbf{u} \in L^2(0,T;V) \cap L^2(0,T;H^2(\Omega))$, we have, for some constant $C>0$,

$$\|\mathbf{u}-\mathbf{U}\|^2_{C([0,T];[L^2(\Omega)]^2)} \leqslant C(\Delta t + h^2) \tag{114}$$

Proof

For $n \geqslant 1$, let $\mathbf{E}^n \equiv (\mathbf{u}_h^n - \mathbf{u}_h^{n-1})/\Delta t$. First consider the case $n=1$. Letting $\mathbf{v}_h = \mathbf{E}^1$, and recalling the Lipschitz continuity and monotonicity of A in Propositions 1 and 2,

$$\begin{aligned}
\|\mathbf{E}^1\|^2_{0,2} &\leqslant \|\mathbf{E}^1\|^2_{0,2} + \left(A\mathbf{u}_h^1 - A\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t} \right) \\
&= (\mathbf{f}^1, \mathbf{E}^1) - \left(A\mathbf{u}_h^0, \frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t} \right) \\
&\leqslant (\|\mathbf{f}^1 - \mathbf{f}^0\|_{0,2} + \|\mathbf{f}^0\|_{0,2} + M\|\mathbf{u}_h^0\|)\|\mathbf{E}^1\|_{0,2}
\end{aligned} \tag{115}$$

As $\mathbf{f}$ is Lipschitz continuous in time, we get

$$\|\mathbf{E}^1\|_{0,2} \leqslant C\,(\Delta t + \|\mathbf{f}^0\|_{0,2} + \|\mathbf{u}^0\|) \tag{116}$$

For $n>1$, we have

$$(\mathbf{E}^n, \mathbf{v}_h) + (A\mathbf{u}_h^n, \mathbf{v}_h) = (\mathbf{f}^n, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in V_h \tag{117}$$

and

$$(\mathbf{E}^{n-1}, \mathbf{v}_h) + (A\mathbf{u}_h^{n-1}, \mathbf{v}_h) = (\mathbf{f}^{n-1}, \mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in V_h \tag{118}$$

Setting $\mathbf{v}_h = \mathbf{E}^n$ and subtracting (118) from (117), and again using Propositions 1 and 2, we have

$$\begin{aligned}
\|\mathbf{E}^n\|^2_{0,2} &\leqslant \|\mathbf{E}^n\|^2_{0,2} + \left(A\mathbf{u}_h^n - A\mathbf{u}_h^{n-1}, \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t} \right) \\
&= (\mathbf{f}^n - \mathbf{f}^{n-1}, \mathbf{E}^n) + (\mathbf{E}^{n-1}, \mathbf{E}^n) \\
&\leqslant (\|\mathbf{f}^n - \mathbf{f}^{n-1}\|_{0,2} + \|\mathbf{E}_h^{n-1}\|_{0,2})\|\mathbf{E}^n\|_{0,2}
\end{aligned} \tag{119}$$

and so

$$\left\| \frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t} \right\|_{0,2} = \|\mathbf{E}^n\|_{0,2} \leqslant C \tag{120}$$

where the constant C depends on $T, \mathbf{f}^0, \mathbf{u}^0$ and the Lipschitz constant of $\mathbf{f}$.

In view of Propositions 1 and 2, we see that, using arguments similar to those in Reference [12], $A\mathbf{u}_h^n$ may be extended to a bounded linear operator on $L^2(\Omega)$ and hence the sequence $\{\mathbf{u}_h^n\}$ is uniformly bounded. Consequently, the functions $\mathbf{U}(t)$ and $\mathbf{U}_h^n(t)$ defined above and their corresponding time derivatives are all uniformly bounded in the $L^2(\Omega)$ norm with respect to n and t.

Note that for $t_{n-1} < t \leqslant t_n$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{U}(t) = \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{W}(t) = \frac{\mathbf{w}^n - \mathbf{w}^{n-1}}{\Delta t} \tag{121}$$

For $t_n < t \leqslant t_{n+1}$,

$$\begin{aligned}
&\left(\frac{\mathbf{w}^n - \mathbf{w}^{n-1}}{\Delta t} - \frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}}{\Delta t}, \mathbf{W}(t) - \mathbf{U}(t)\right) \\
&\quad = \left(\frac{\mathrm{d}}{\mathrm{d}t}W(t) - \frac{\mathrm{d}}{\mathrm{d}t}U(t), \mathbf{W}(t) - \mathbf{U}(t)\right) \\
&\quad = \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{W}(t) - \mathbf{W}_h^n\right) + \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{W}_h^n - \mathbf{u}_h^n(t)\right) \\
&\qquad + \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{u}_h^n(t) - \mathbf{U}(t)\right)
\end{aligned} \tag{122}$$

Now, by (113), (121), (122), and the monotonicity of A,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{W}(\mathbf{t}) - \mathbf{U}(\mathbf{t})\|_{0,2}^2 \\
&\quad = \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{W}(t) - \mathbf{U}(t)\right) \\
&\quad = \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{W}(t) - \mathbf{W}_h^n\right) - (A\mathbf{W}_h^n - A\mathbf{u}_h^n, \mathbf{W}_h^n - \mathbf{u}_h^n) \\
&\qquad + \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{u}_h^n - \mathbf{U}(t)\right) \\
&\quad \leqslant \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{W}(t) - \mathbf{W}_h^n\right) + \left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{u}_h^n - \mathbf{U}(t)\right)
\end{aligned} \tag{123}$$

As the derivative $\mathrm{d}\mathbf{U}/\mathrm{d}t$ is uniformly bounded in the $L^2(\Omega)$ norm and a similar result holds for $\mathrm{d}\mathbf{W}/\mathrm{d}t$, we have

$$\begin{aligned}
&\left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{W}(t) - \mathbf{W}_h^n\right) \\
&\quad \leqslant C\|\mathbf{W}(t) - \mathbf{W}_h^n\|_{0,2} \\
&\quad = \left\|\frac{t-t_n}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^n) + \mathbf{u}^n - \mathbf{W}_h^n\right\|_{0,2} \\
&\quad \leqslant C\left(|t_n - t| \left\|\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t}\right\|_{0,2} + \|\mathbf{u}^n - \mathbf{W}_h^n\|_{0,2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C(\Delta t + \|\mathbf{u}^n - \mathbf{W}_h^n\|_{0,2}) \\
&\leqslant C(\Delta t + h^2) \qquad (124)
\end{aligned}
$$

Also

$$
\begin{aligned}
\left(\frac{\mathrm{d}(\mathbf{W}-\mathbf{U})}{\mathrm{d}t}, \mathbf{u}_h^n - \mathbf{U}(t)\right) &\leqslant C\|\mathbf{u}_h^n - \mathbf{U}(t)\|_{0,2} \\
&= \left\|\frac{t-t_n}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^n)\right\|_{0,2} \\
&\leqslant C\left(|t_n - t|\left\|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}\right\|_{0,2}\right) \\
&\leqslant C\,\Delta t \qquad (125)
\end{aligned}
$$

Thus from (123)–(125) we have, for $t_n < t \leqslant t_{n+1}$, $n = 0, 1, \ldots, N-1$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{W}(t) - \mathbf{U}(t)\|_{0,2}^2 \leqslant C(\Delta t + h^2) \qquad (126)$$

Integrating over $[0, t]$, we have

$$\|\mathbf{W}(t) - \mathbf{U}(t)\|_{0,2}^2 \leqslant \|\mathbf{u}_0 - \mathbf{u}_{0h}\|_{0,2}^2\| + C(\Delta t + h^2) \qquad (127)$$

Consequently, from the above result and (111),

$$
\begin{aligned}
\|\mathbf{u}(t) - \mathbf{U}(t)\|_{0,2}^2 &\leqslant 2\|\mathbf{u}(t) - \mathbf{W}(t)\|_{0,2}^2 + 2\|\mathbf{W}(t) - \mathbf{U}(t)\|_{0,2}^2 \\
&\leqslant C(\Delta t + h^2) \qquad (128)
\end{aligned}
$$

and, taking the supremum over $[0, T]$, the theorem is proved.

9. CONCLUDING REMARKS

Here we consider several models for generalized Newtonian fluids and prove optimal *a priori* estimates under appropriate assumptions for an interesting class of divergence free elements that are computationally appealing for these models. These estimates and the *a posteriori* estimates that we also give here provide the theoretical foundation for applying finite elements to flow applications involving these generalized Newtonian fluid models.

To our knowledge, no corresponding three-dimensional PSH element has been constructed and it is not clear if such element even exists. If a piecewise quadratic C^1 element with small support could be found, then the results derived in this paper would remain valid, with some suitable modification for the *a posteriori* estimates.

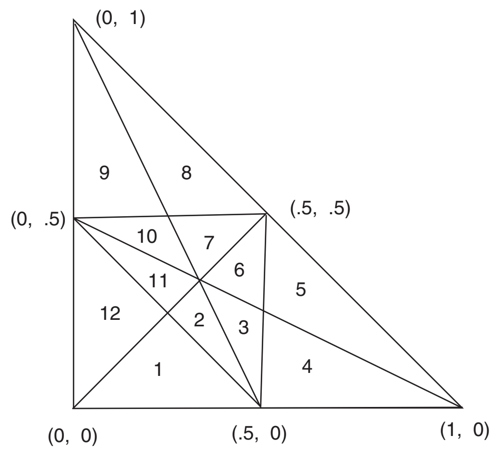

Figure 2. Numbering of subtriangles on reference element.

APPENDIX A: POWELL–SABIN–HEINDL ELEMENT BASIS FUNCTIONS

Here we list the Powell–Sabin–Heindl 12-split (2) element basis functions over the reference triangle with vertices $a_0=(0,0), a_1=(1,0), a_2=(0,1)$. By taking the curl of these functions, the basis functions for the divergence free Powell–Sabin–Heindl element may be constructed (Figure 2).

1. Basis function corresponding to vertex a_0:

Sub$\triangle$	Polynomial
1	$-2x^2-2y^2+1$
2	$-6x^2-8xy-6y^2+4x+4y$
3	$2x^2-4y^2-4x+2$
4	$2x^2-4y^2-4x+2$
5	$4(x+y-1)^2$
6	$4(x+y-1)^2$
7	$4(x+y-1)^2$
8	$4(x+y-1)^2$
9	$-4x^2+2y^2-4y+2$
10	$-4x^2+2y^2-4y+2$
11	$-6x^2-8xy-6y^2+4x+4y$
12	$-2x^2-2y^2+1$

2. Basis function corresponding to vertex a_1:

Sub△	Polynomial
1	$2x^2$
2	$4x^2+4xy+2y^2-2x-2y+\frac{1}{2}$
3	$y^2+2x-\frac{1}{2}$
4	$-2x^2+y^2+4x-1$
5	$-3x^2-4xy-3y^2+6x+4y-2$
6	$-x^2-4xy-3y^2+4x+4y-\frac{3}{2}$
7	$-x^2-4xy-3y^2+4x+4y-\frac{3}{2}$
8	$-x^2-4xy-y^2+4x+2y-1$
9	$3x^2$
10	$3x^2-2y^2+2y-\frac{1}{2}$
11	$4x^2+4xy+2y^2-2x-2y+\frac{1}{2}$
12	$2x^2$

3. Basis function corresponding to vertex a_2:

Sub△	Polynomial
1	$2y^2$
2	$2x^2+4xy+4y^2-2x-2y+\frac{1}{2}$
3	$-2x^2+3y^2+2x-\frac{1}{2}$
4	$3y^2$
5	$-x^2-4xy-y^2+2x+4y-1$
6	$-3x^2-4xy-y^2+4x+4y-\frac{3}{2}$
7	$-3x^2-4xy-y^2+4x+4y-\frac{3}{2}$
8	$-3x^2-4xy-3y^2+4x+6y-2$
9	x^2-2y^2+4y-1
10	$x^2+2y-\frac{1}{2}$
11	$2x^2+4xy+4y^2-2x-2y+\frac{1}{2}$
12	$2y^2$

4. Basis function corresponding to x-derivative at vertex a_0:

Sub△	Polynomial
1	$-\frac{3}{2}x^2 - y^2 + x$
2	$-\frac{3}{2}x^2 - y^2 + x$
3	$-\frac{3}{2}x^2 - y^2 + x$
4	$\frac{1}{2}x^2 - y^2 - x + \frac{1}{2}$
5	$(x + y - 1)^2$
6	$-x^2 + 2xy + y^2 - 2y + \frac{1}{2}$
7	$\frac{1}{2}(2y - 1)^2$
8	0
9	0
10	$\frac{1}{2}(2y - 1)^2$
11	$-\frac{1}{2}(x + 4y - 2)x$
12	$-\frac{1}{2}(x + 4y - 2)x$

5. Basis function corresponding to y-derivative at vertex a_0:

Sub△	Polynomial
1	$-\frac{1}{2}(4x + y - 2)y$
2	$-\frac{1}{2}(4x + y - 2)y$
3	$\frac{1}{2}(2x - 1)^2$
4	0
5	0
6	$\frac{1}{2}(2x - 1)^2$
7	$x^2 + 2xy - y^2 - 2x + \frac{1}{2}$
8	$(x + y - 1)^2$
9	$-x^2 + \frac{1}{2}y^2 - y + \frac{1}{2}$
10	$-x^2 - \frac{3}{2}y^2 + y$
11	$-x^2 - \frac{3}{2}y^2 + y$
12	$-x^2 - \frac{3}{2}y^2 + y$

6. Basis function corresponding to x-derivative at vertex a_1:

Sub△	Polynomial
1	$-\frac{1}{2}x^2$
2	$-x^2 - xy - \frac{1}{2}y^2 + \frac{1}{2}x + \frac{1}{2}y - \frac{1}{8}$
3	$-\frac{1}{4}y^2 - \frac{1}{2}x + \frac{1}{8}$
4	$\frac{3}{2}x^2 - \frac{1}{4}y^2 - 2x + \frac{1}{2}$
5	$\frac{7}{4}x^2 + xy + \frac{3}{4}y^2 - \frac{5}{2}x - y + \frac{3}{4}$
6	$\frac{1}{4}x^2 + xy + \frac{3}{4}y^2 - x - y + \frac{3}{8}$
7	$\frac{1}{4}x^2 + xy + \frac{3}{4}y^2 - x - y + \frac{3}{8}$
8	$\frac{1}{4}x^2 + xy + \frac{1}{4}y^2 - x - \frac{1}{2}y + \frac{1}{4}$
9	$-\frac{3}{4}x^2$
10	$-\frac{3}{4}x^2 + \frac{1}{2}y^2 - \frac{1}{2}y + \frac{1}{8}$
11	$-x^2 - xy - \frac{1}{2}y^2 + \frac{1}{2}x + \frac{1}{2}y - \frac{1}{8}$
12	$-\frac{1}{2}x^2$

7. Basis function corresponding to y-derivative at vertex a_1:

Sub△	Polynomial
1	0
2	$\frac{3}{8}(2x + 2y - 1)^2$
3	$\frac{1}{2}x^2 + 2xy + \frac{5}{4}y^2 - \frac{1}{2}x - y + \frac{1}{8}$
4	$\frac{1}{4}(8x + 5y - 4)y$
5	$-\frac{3}{4}x^2 - xy - \frac{7}{4}y^2 + \frac{3}{2}x + 2y - \frac{3}{4}$
6	$-\frac{1}{4}x^2 - xy - \frac{7}{4}y^2 + x + 2y - \frac{5}{8}$
7	$-\frac{1}{4}x^2 - xy - \frac{7}{4}y^2 + x + 2y - \frac{5}{8}$
8	$-\frac{1}{4}x^2 - xy - \frac{1}{4}y^2 + x + \frac{1}{2}y - \frac{1}{4}$
9	$\frac{3}{4}x^2$
10	$\frac{3}{4}x^2 - \frac{3}{2}y^2 + \frac{3}{2}y - \frac{3}{8}$
11	$\frac{3}{8}(2x + 2y - 1)^2$
12	0

8. Basis function corresponding to x-derivative at vertex a_2:

Sub△	Polynomial
1	0
2	$\frac{3}{8}(2x+2y-1)^2$
3	$-\frac{3}{2}x^2+\frac{3}{4}y^2+\frac{3}{2}x-\frac{3}{8}$
4	$\frac{3}{4}y^2$
5	$-\frac{1}{4}x^2-xy-\frac{1}{4}y^2+\frac{1}{2}x+y-\frac{1}{4}$
6	$-\frac{7}{4}x^2-xy-\frac{1}{4}y^2+2x+y-\frac{5}{8}$
7	$-\frac{7}{4}x^2-xy-\frac{1}{4}y^2+2x+y-\frac{5}{8}$
8	$-\frac{7}{4}x^2-xy-\frac{3}{4}y^2+2x+\frac{3}{2}y-\frac{3}{4}$
9	$\frac{1}{4}(5x+8y-4)x$
10	$\frac{5}{4}x^2+2xy+\frac{1}{2}y^2-x-\frac{1}{2}y+\frac{1}{8}$
11	$\frac{3}{8}(2x+2y-1)^2$
12	0

9. Basis function corresponding to y-derivative at vertex a_2:

Sub△	Polynomial
1	$-\frac{1}{2}y^2$
2	$-\frac{1}{2}x^2-xy-y^2+\frac{1}{2}x+\frac{1}{2}y-\frac{1}{8}$
3	$\frac{1}{2}x^2-\frac{3}{4}y^2-\frac{1}{2}x+\frac{1}{8}$
4	$-\frac{3}{4}y^2$
5	$\frac{1}{4}x^2+xy+\frac{1}{4}y^2-\frac{1}{2}x-y+\frac{1}{4}$
6	$\frac{3}{4}x^2+xy+\frac{1}{4}y^2-x-y+\frac{3}{8}$
7	$\frac{3}{4}x^2+xy+\frac{1}{4}y^2-x-y+\frac{3}{8}$
8	$\frac{3}{4}x^2+xy+\frac{7}{4}y^2-x-\frac{5}{2}y+\frac{3}{4}$
9	$-\frac{1}{4}x^2+\frac{3}{2}y^2-2y+\frac{1}{2}$
10	$-\frac{1}{4}x^2-\frac{1}{2}y+\frac{1}{8}$
11	$-\frac{1}{2}x^2-xy-y^2+\frac{1}{2}x+\frac{1}{2}y-\frac{1}{8}$
12	$-\frac{1}{2}y^2$

10. Basis function corresponding to mid-edge point $(\frac{1}{2}, \frac{1}{2})$:

Sub△	Polynomial
1	0
2	$\frac{1}{4}\sqrt{2}\,(2x+2y-1)^2$
3	$-\frac{1}{4}\sqrt{2}(2x+\sqrt{2}y-1)(2x-\sqrt{2}y-1)$
4	$\frac{1}{2}\sqrt{2}y^2$
5	$-\frac{1}{2}\sqrt{2}(x+3y-1)(x+y-1)$
6	$-\frac{1}{4}\sqrt{2}(6x^2+8xy+6y^2-8x-8y+3)$
7	$-\frac{1}{4}\sqrt{2}(6x^2+8xy+6y^2-8x-8y+3)$
8	$-\frac{1}{2}\sqrt{2}(x+y-1)(3x+y-1)$
9	$\frac{1}{2}\sqrt{2}x^2$
10	$\frac{1}{8}\sqrt{2}(2x+2\sqrt{2}y-\sqrt{2})(2x-2\sqrt{2}y+\sqrt{2})$
11	$\frac{1}{4}\sqrt{2}(2x+2y-1)^2$
12	0

11. Basis function corresponding to mid-edge point $(0, \frac{1}{2})$:

Sub△	Polynomial
1	y^2
2	$-2x^2-4xy-y^2+2x+2y-\frac{1}{2}$
3	$\frac{1}{2}\,(2x-1)^2$
4	0
5	0
6	$\frac{1}{2}\,(2x-1)^2$
7	$x^2+2xy-y^2-2x+\frac{1}{2}$
8	$(x+y-1)^2$
9	$-(3x+2y-2)x$
10	$-3x^2-2xy-2y^2+2x+2y-\frac{1}{2}$
11	$-3x^2-2xy-2y^2+2x+2y-\frac{1}{2}$
12	$-(x-2y)x$

12. Basis function corresponding to mid-edge point $(\frac{1}{2}, 0)$:

Sub△	Polynomial
1	$(2x - y)y$
2	$-2x^2 - 2xy - 3y^2 + 2x + 2y - \frac{1}{2}$
3	$-2x^2 - 2xy - 3y^2 + 2x + 2y - \frac{1}{2}$
4	$-(2x + 3y - 2)y$
5	$(x + y - 1)^2$
6	$-x^2 + 2xy + y^2 - 2y + \frac{1}{2}$
7	$\frac{1}{2}(2y - 1)^2$
8	0
9	0
10	$\frac{1}{2}(2y - 1)^2$
11	$-x^2 - 4xy - 2y^2 + 2x + 2y - \frac{1}{2}$
12	x^2

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